# How many edges can be shared by N square tiles on a board?

**e-lab Research Report - April 2008**

Thierry Benoist

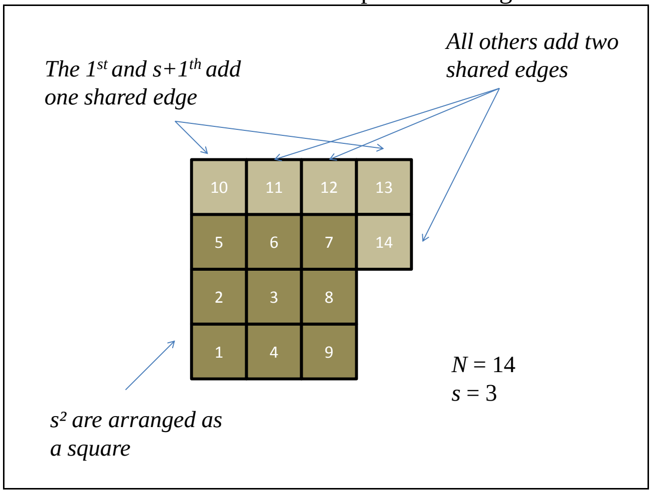
e-lab - Bouygues SA – 32 avenue Hoche 75008 Paris
**tbenoist@bouygues.com**

## 1. Introduction

In this note we build an arrangement of N square tiles maximizing the number of shared edges and we present a formula returning this optimal value for any N. This question arises when considering edge-matching puzzles. Computing this maximum provides a necessary condition for the feasibility of a puzzle.
More precisely we are considering the following problem. Given an integer $N$, we want to define a set $S$ of $N$ pairwise different "coordinates" $(x,y)$ maximizing the number of pairs $((x,y),(x',y')) \in S^2$ such that ($x = x'$ and $y = y'+1$) or ($x = x'+1$ and $y = y'$).

## 2. Solution

In what follows, we propose a simple arrangement of tiles and prove its optimality by recurrence.
With $s = \lfloor\sqrt{N}\rfloor$, we can arrange $s^2$ tiles in a square of side $s$. In such a square $2s(s-1)$ edges are shared[1]. The remaining $N$-$s^2$ tiles can be positioned around this square, starting next to a corner of the square. Proceeding like this, each added tile adds two shared edges except for the first and the $s$+1th one. The figure below shows such an arrangement for N=14 (tiles have been numbered so as to represent arrangements for N<14 as well).




Finally we obtain the following formula, where all cases are covered in the second term since n-s² cannot exceed $2s$, otherwise we would have $N \geq (s+1)^2$ what would contradict the definition of $s$.

$$with\ s = \lfloor\sqrt{N}\rfloor, h(N) = 2s(s-1) + \begin{cases} 0 & if \quad n - s^2 = 0 \\ 2(n-s^2)-1 & if \quad 0 < n - s^2 \leq s \\ 2(n-s^2)-2 & if \quad s < n - s^2 \leq 2s \end{cases}$$

Now we can prove that this arrangement is optimum.

## 3. Proof

***Lemma 1.*** *For all integer s, and for both δ=0 and δ=1, any rectangle of area larger or equal to s(s+δ) whose larger and smaller sides are not s+δ and s, respectively has a perimeter strictly larger than 2(2s+δ).*
**Proof:** Assume that we have a rectangle of larger and smaller sides X and Y contradicting the above lemma.

[1] $s$-1vertical lines of length $s$ and $s$-1 horizontal lines of length $s$

That is to say $X=s+\delta+\alpha$, $Y=s-\alpha-\varepsilon$ with $\varepsilon \geq 0$ (since for look for a smaller or equal perimeter). By definition of X as the larger side, $\alpha$ cannot be negative, and since we look for a different rectangle $\alpha$ cannot be 0, which meant that $\alpha\geq1$. If we compute the area of the rectangle we obtain

$$XY = (s+\delta+\alpha)(s-\alpha-\varepsilon) = (s+\delta)s - \varepsilon s - \delta(\alpha+\varepsilon) - \alpha\varepsilon - \alpha^2$$

This area is strictly smaller than $(s+\delta)s$ because all terms after $(s+\delta)s$ are negative and $-\alpha^2$ is strictly negative. This contradictions proves that this X,Y rectangle we smaller perimeter does not exists. Hence the lemma is true.□

***Lemma 2.*** *When $N=s(s+\delta)$ with $\delta\in\{0,1\}$, h(N) is optimal and the described solution is unique (any other tile arrangement is strictly worse).*

**Proof:** For any arrangement, the rightmost tile of each row has no right neighbor and the topmost tile of each column as no top neighbor. Therefore the number of shared edges cannot exceed 2N-X-Y where X and Y are the number of columns and rows, respectively. From lemma 1, we conclude that any arrangement of these N tiles different from the rectangle $(s,s+\delta)$, fits necessarily in a rectangle of larger perimeter ($X+Y>2s+\delta$). Finally the number of shared edges in this different arrangement is strictly smaller than $2s(s+\delta)-2s-\delta=h(N)$. □

***Proposition.*** *The maximum number of shared edges for all integer N is h(N).*

**Proof.** First, $h(N)$ is optimal for $N$=1. Now let us assume that $h(N-1)$ is optimal.

Here two cases must be distinguished.

i) If $N$ is not equal to $s^2+1$ nor to $s(s+1)+1$.
Then the bottom leftmost tile has at most two shared edges. As for the remaining $N$-1 tiles they cannot shared more than $h(N-1)$ edges (our recurrence hypothesis). Therefore the number of shared edges cannot be greater than $2+h(N-1)=h(N)$. Hence $h(N)$ is optimal.

ii) If $N$ is equal to $s^2+1$ or to $s(s+1)+1$.
If we consider all tiles but the bottom leftmost one, they cannot share more than $h(N-1)$ edges. More precisely from lemma 2 we know that they are either arranged in a rectangular shape or share less than $h(N-1)-1$ edges.
   a. If they form and $s(s+\delta)$ rectangle, then the remaining tile cannot have more than one shared edge, and the total is bounded by $h(N-1)+1 = h(N)$
   b. If they share less than $h(N-1)-1$ edges, then even if the bottom leftmost tile has two shared edges, the total does not execeed $h(N-1)-1+2 = h(N)$.

Finally, is $h(N-1)$ is optimal then $h(N)$ is optimal. Since $h(1)=0$ is obviously optimal we conclude that $h(N)$ is optimal for all $N$. □

## 4. Conclusion

The property proven in this note can be used as a necessary condition for the feasibility of an edge-matching puzzle. If a color $c$ appears on $E$ edges dispatched on $N$ tiles, then $E \leq 2h(N)$ is a necessary condition for the feasibility of the puzzle, because each matching pair of c-colored edges is a shared edge of the tiles arrangement. For instance 10 colored edges dispatched on 4 tiles cannot be arranged properly. This condition is not sufficient. For instance positioning 4 tiles with 8 $c$-colored edges is only possible if no tile has two opposite $c$-colored edges.

This rule also applies for partially instantiated boards. With $E'$ the number of already positioned $c$-colored edges with no matching tile yet, then $E \leq 2h(N) + E'$ is a necessary condition. In other words each shared edge can absorb two $c$-colored edges and each of the $E'$ "open" edge can absorb one c-colored edge.

# References

**E.D. Demaine, M. L. Demaine (2007)**. *Jigsaw Puzzles, Edge Matching, and Polyomino Packing: Connections and Complexity.* In Graphs and Combinatorics vol 23, pp 195-208, Springer Japan.